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Resonance formation of Kirkwood gaps and asteroid clusters

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Abstract. A mathematical description of the Kirkwood gap in the 2/1 resonance with Jupiter is discussed in detail, providing a new insight into a traditionally controversial issue. The discussion is further extended to demonstrate how, under slightly different circumstances, the same kind of resonance can create the opposite effect of asteroid clustering.

It has been shown [1] and recently re-derived in more detail [5] that the behaviour of an asteroid in the 2/1 resonance with Jupiter (subject to no other perturbing forces) can be described, to a simple but adequate approximation, by the following set of differential equations:

$$\vartheta' = -\frac{c\varepsilon}{\beta}\cos(\vartheta) - 2(P-1) \tag{1a}$$

$$\beta' = -c\varepsilon\sin(\vartheta) \tag{1b}$$

$$P' = 12c\varepsilon\beta\sin(\vartheta) \tag{1c}$$

where P is the asteroid's orbital period (by the choice of units equal to one in the exact 2/1 resonance), ε is Jupiter's mass (relative to the Sun's), $2\beta/(1+\beta^2)$ is the asteroid's eccentricity, ϑ is its aphelion's angular distance from conjunction (the 'resonance variable' of [3]), c equals to

$$\tfrac{9}{16}F\left(\tfrac{1}{2},\tfrac{5}{2},3;2^{-4/3}\right) + \tfrac{5}{64}2^{-4/3}F\left(\tfrac{3}{2},\tfrac{7}{2},4;2^{-4/3}\right) \simeq 0.75 \tag{2}$$

(correcting a 'misprint' in [5]), and the independent variable is the asteroid's 'modified time' ($\simeq$ one half of the asteroid's eccentric anomaly).

From equations (1b) and (1c) it easily follows that $P + 6\beta^2$ is a constant, which can be used to eliminate P and rewrite (1a) as follows:

$$\vartheta' = -\frac{c\varepsilon}{\beta}\cos(\vartheta) + 12\beta^2 + K \tag{3}$$

where

$$K = 2 - 2P_0 - 12\beta_0^2 \tag{4}$$

(P_0 and β_0 being the initial values). Equations (3) and (1b) further imply that

$$-2c\varepsilon\cos(\vartheta)\beta + K\beta^2 + 6\beta^4 \tag{5}$$

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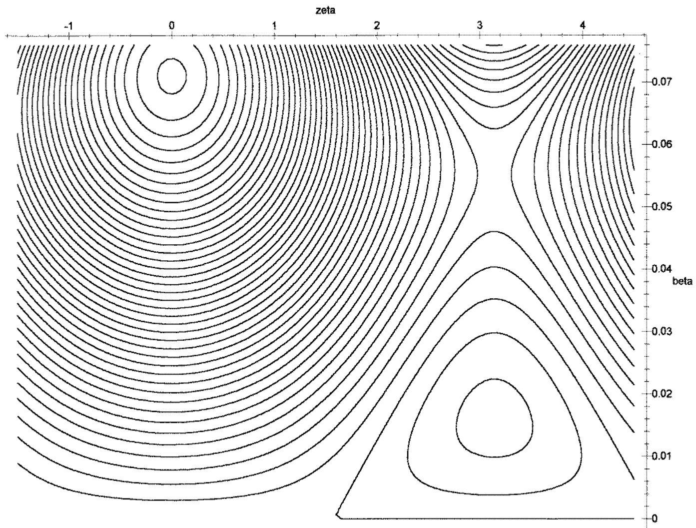

Figure 1. Contour plot of (5) for $K = -0.05$ with 'zeta' being the resonance variable and 'beta' $\simeq$ half the asteroid's eccentricity. The nature of possible solutions to (1) is apparent.

is yet another constant, clearly indicating that the solution to (1) is periodic when in the basin of either one ($K > -0.0357$) or two ($K < -0.0357$) of the system's centres. These are found by making the right-hand sides of (3) and (1*b*) identically equal to zero. Some of this is illustrated in figure 1 which displays the contour plot of (5) with $K = -0.05$ with four distinct regions: two with the aphelion circulating, one each with the aphelion or perihelion librating around conjunction.

The above analysis needs to be augmented when the asteroids are also subject to various non-conservative forces, of which the most prominent is the one caused by collisions with other orbiting bodies [1]. Slightly at variance with the reference just cited, we argue that this effect is proportional to the speed of the asteroid, *relative* to the *average* speed of particles orbiting the sun at the asteroid's instantaneous location. The average speed is established based on *circular* orbits (thus, the asteroid at aphelion would be more likely to be hit from behind, at the perihelion from the front). This 'Kepler shear' corresponds to a force in the direction of the asteroid's velocity and proportional to $\beta\cos[2(s - s_0)]$, where $2(s - s_0)$ is the eccentric anomaly, set to zero at aphelion. The contribution of such a force to our differential equations (1) is an extra $-C\beta$ term on the right-hand side of (1*b*), C being a small constant (related to the asteroid's size). This term is sufficient to damp the original solution to one of its centres (which have now become spiral foci of [4]), found by solving (7*a*) and $\sin(\vartheta_{\mathrm{f}}) = 0$. The location of these foci is plotted (in terms of the resulting $P \stackrel{\mathrm{def}}{\equiv} P_{\mathrm{f}}$, obtained from $K = 2 - 2P_{\mathrm{f}} - 12\beta_{\mathrm{f}}^2$) against K in figure 2. The creation of a 1.3% gap is quite apparent. By analysing contour plots analogous to figure 1 with different values of K, and by numerical experimentation with the modified equations (1), it becomes clear that asteroids in the $K < -0.0357$ regime would normally be trapped by the upper (low-eccentricity) branch of figure 2, with the perihelion locked in with conjunction.

The linearized version of (3) and (1*b*) (with the extra $-C\beta$ term) looks as follows:

$$\begin{bmatrix} \vartheta - \vartheta_{\mathrm{f}} \\ \beta - \beta_{\mathrm{f}} \end{bmatrix}' = \begin{bmatrix} -C & 24\beta_{\mathrm{f}} \mp c\varepsilon/\beta_{\mathrm{f}}^2 \\ \pm c\varepsilon & -C \end{bmatrix} \cdot \begin{bmatrix} \vartheta - \vartheta_{\mathrm{f}} \\ \beta - \beta_{\mathrm{f}} \end{bmatrix} + \cdots \qquad (6)$$

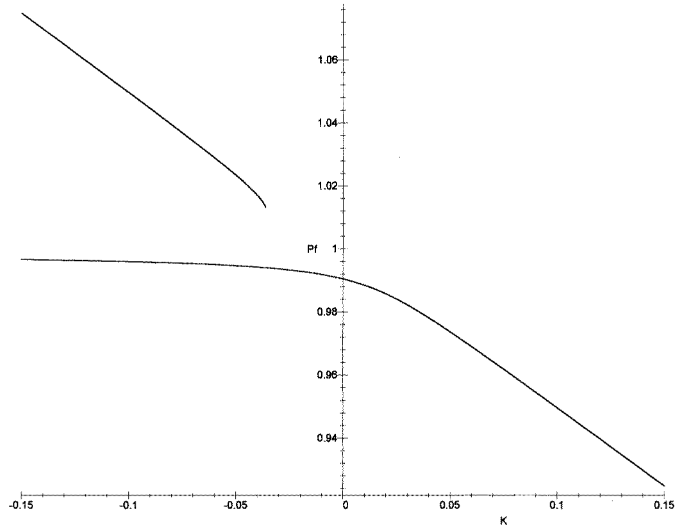

Figure 2. Stable values of (1) in terms of P_f (the damped value of the orbital period) plotted against K representing initial conditions.

where the f subscript implies a fixed value of (3) and (1b), obtained from

$$\pm c\varepsilon/\beta_f + 12\beta_f^2 + K = 0 \tag{7a}$$

and

$$\sin(\vartheta_f) = -\frac{C\beta_f}{c\varepsilon} \tag{7b}$$

(the plus and minus signs implying the high- and low-eccentricity solutions, respectively). In equation (7a) we assumed that C was small enough to allow $\cos(\vartheta_f) \simeq \mp 1$. Note that (7$a$), being a cubic equation for β_f, has either one real root or three, one of the latter corresponding to a saddle point and as such discarded (the eligible roots were plotted in figure 2). From the two eigenvalues of the matrix in (6) we can clearly see that the ϑ and β (and consequently P)-libration/oscillation will be damped by a factor of $\exp(-Cs)$ to one of the stable points of figure 1. Furthermore, based on the same matrix, one can compute the asymptotic frequency of these oscillations (a non-trivial function of the inseparable $c\varepsilon$ and K), which we display, relative to the asteroid's orbital frequency, in figure 3.

The linearized treatment of the previous paragraph was carried out assuming that $P+6\beta^2$ remained constant. This, of course, is now valid only approximately. In the actual solution, the damped value of P will no longer stay constant, but will slowly decrease according to

$$P' = -12C\beta_f^2 . \tag{8}$$

This drift in P will tend to modify the original density of the asteroids, but not very noticeably except in the actual gap. When an asteroid with $P_0 > 1.013$ reaches the gap, the focus in which it had stabilized disappears, and the asteroid is forced to make a quick transition to the only remaining stable point, which lies *below* the gap. The actual transition is illustrated in figure 4 (using an exaggerated value of $C = 0.0005$ to speed up the process). This explains why hardly any asteroids are found inside the gap itself.

The above analysis applies (qualitatively the coefficients will differ and ϑ changes to $\vartheta + \pi/2$ for r odd) to any other $r/(r-1)$ resonance. Why then do some of these resonances,

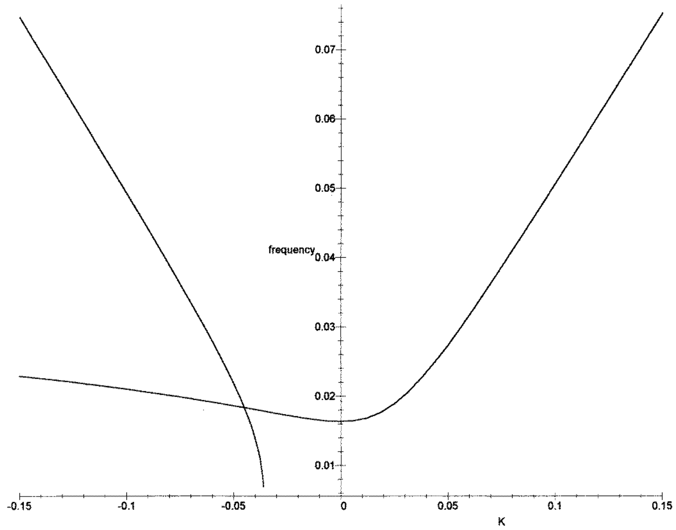

Figure 3. Asymptotic (near-stable-point) frequency of orbital period and eccentricity, relative to the asteroid's orbital frequency.

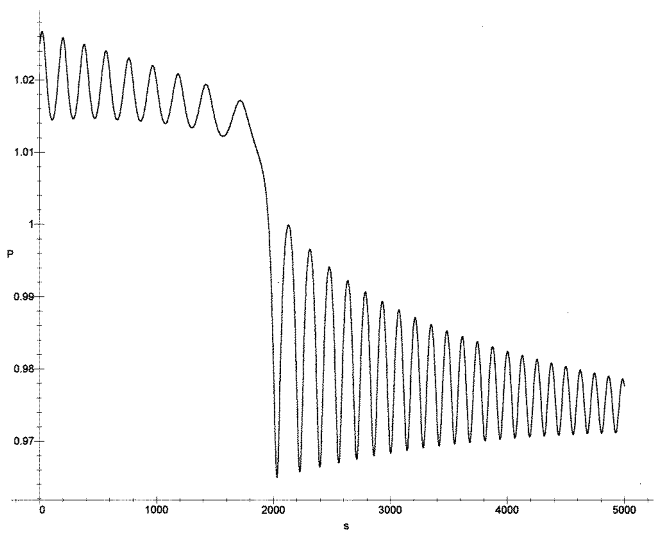

Figure 4. Sample solution to (1), with an extra -0.0005β term in (1b) causing damping. The orbital period is displayed against modified time.

contrary to our explanation so far, tend to cluster asteroids around the commensurability, depleting the surrounding region? The mystery lies in yet another extra term needed in (1) to make our mathematical description fully realistic. Due to many possible extra forces, some of them conservative, some dissipative, the asteroids' orbital period P will be perturbed in either a periodic manner (the effect of Saturn and other planets) or in a systematic way (drag, Sun's tidal forces, meteorite showers). We assume that the strongest one, be it systematic or cyclical, is changing so slowly that, for a given time, it can be considered constant. We thus need to add an appropriate κ to the right-hand side of (1c) and repeat our analysis of the equations. When $\kappa < 0$ (energy loss), qualitatively nothing changes, only the P's downward drift will increase slightly (in the spirit of the subsequent discussion we may even see this κ as the driving force of the gap formation, with C being a small stabilizing subsidiary). On the other hand, when $\kappa > 0$ (energy boost), *however small*, the direction of the P-drift is reversed. This time, there is a single and rather trivial *stable* solution to (1), given by

$$P_{\mathrm{f}} = 1 - c\varepsilon\sqrt{\frac{3C}{\kappa}\left(1 - \frac{\kappa C}{12c^2\varepsilon^2}\right)} \approx 1 \qquad (9a)$$

$$\beta_{\mathrm{f}} = \sqrt{\frac{\kappa}{12C}} \qquad (9b)$$

and

$$\sin(\vartheta_{\mathrm{f}}) = -\frac{\sqrt{\kappa C}}{\sqrt{12}c\varepsilon}\,. \qquad (9c)$$

Thus, the asteroids below and inside the gap will eventually get trapped at its lower edge, the ones above the gap will drift away, to be trapped by the next available resonance. This explains the creation of asteroid clusters.

In conclusion, either a gap or a cluster will form in a $r/(r-1)$ resonance, depending on whether the resonance is located in a region with a negative or a positive κ, respectively (κ itself may change in time when all perturbations due to the rest of the solar system are fully accounted for, but such an analysis is beyond the scope of this paper). This is because a resonance tends to act as a 'semi-permeable membrane', letting the asteroids through (and very quickly so) in one direction only. And, as a simple demonstration of how it can act as a barrier, we display two solutions to (1) with both extra terms ($C = 0.0005$, and $\kappa = 0.000\,01$) in figure 5. The linearized analysis gives the s-scale asymptotic frequency and

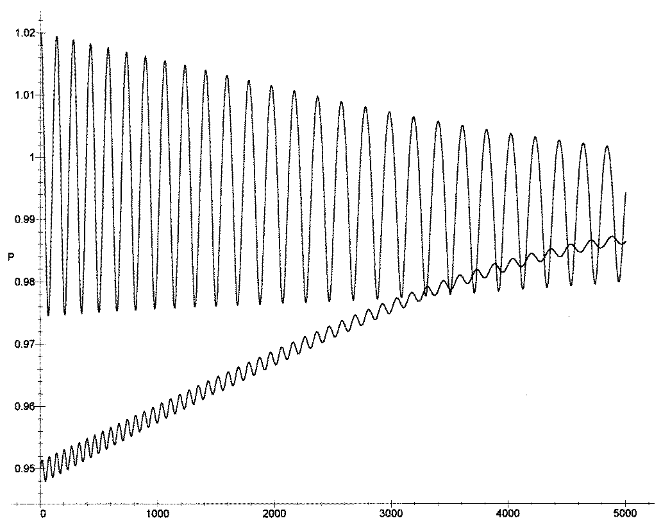

Figure 5. Two sample solutions to (1) with the same damping term as in figure 4 and an extra $+0.000\,01$ term in (1c), restoring stability near resonance.

damping rate (now substantially diminished), as the imaginary and real part, respectively, of the complex roots of the following cubic (assuming $\kappa C \ll 12c^2\varepsilon^2$):

$$\lambda^3 + 2C\lambda^2 + 12c\varepsilon\left(c\varepsilon\frac{C}{\kappa} + \sqrt{\frac{\kappa}{3C}}\right)\lambda + 24c\varepsilon\sqrt{\frac{C\kappa}{3}} = 0\,. \tag{10}$$

One should note that, as the value of c decreases (with increasing distance from Jupiter and higher order of commensurability), both effects (gap and cluster creation) disappear, the former gradually (the actual gaps get too narrow to be observable), the latter rather suddenly as we approach $c\varepsilon = \sqrt{\kappa C}/12(\simeq 0.000\,02$ in our case), where the stable point vanishes, due to (9c), and the resonance ceases to act as a barrier (there is a brief transitional region which allows capture of only some asteroids, depending on initial conditions).

Finally, when the same kind of analysis is applied to the $r/(r-2), \ldots$ resonances (for these, the trigonometric terms of (1) are further multiplied by the same extra power of β [2]), the stable foci disappear (only the high-eccentricity focus can be found for K negative) and the $-C\beta$ term thus becomes inconsequential. A gap still appears (due to a small negative κ), but only when the initial eccentricity is larger than a certain threshold value, resulting in the gap's *partial* clearing. Furthermore, when $\kappa > 0$, these resonances are no longer capable of acting as a permanent barrier (an effect similar to the $\kappa < 0$ gap creation is observed with some initial conditions, only a temporary asteroid's capture with others).

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